UNIVERSITY OF CAMBRIDGE INTERNATIONAL EXAMINATIONS
General Certificate of Education
Advanced Level

FURTHER MATHEMATICS **9231/01**

Paper 1

October/November 2005

3 hours

Additional materials: Answer Booklet/Paper
Graph paper
List of Formulae (MF10)

READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your Centre number, candidate number and name on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.
The use of a calculator is expected, where appropriate.
Results obtained solely from a graphic calculator, without supporting working or reasoning, will not receive credit.
You are reminded of the need for clear presentation in your answers.

This document consists of **5** printed pages and **3** blank pages.

UNIVERSITY *of* CAMBRIDGE
International Examinations

1 Write down the fifth roots of unity. Hence, or otherwise, find all the roots of the equation

$$z^5 = -16 + (16\sqrt{3})\mathrm{i},$$

giving each root in the form $r\mathrm{e}^{\mathrm{i}\theta}$. [

2 The sequence $u_1, u_2, u_3, \ldots$ is such that $u_1 = 1$ and

$$u_{n+1} = -1 + \sqrt{(u_n + 7)}.$$

(i) Prove by induction that $u_n < 2$ for all $n \geqslant 1$. [4]

(ii) Show that if $u_n = 2 - \varepsilon$, where ε is small, then

$$u_{n+1} \approx 2 - \tfrac{1}{6}\varepsilon.$$ [2]

3 The curve C has equation

$$y = \frac{x^2}{x + \lambda},$$

where λ is a non-zero constant. Obtain the equations of the asymptotes of C. [3]

In separate diagrams, sketch C for the cases where

(i) $\lambda > 0$,

(ii) $\lambda < 0$.

[4]

4 Solve the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 24\mathrm{e}^{2x},$$

given that $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 9$ when $x = 0$. [7]

5 In the equation

$$x^3 + ax^2 + bx + c = 0,$$

the coefficients a, b and c are real. It is given that all the roots are real and greater than 1.

(i) Prove that $a < -3$. [1]

(ii) By considering the sum of the squares of the roots, prove that $a^2 > 2b + 3$. [2]

(iii) By considering the sum of the cubes of the roots, prove that $a^3 < -9b - 3c - 3$. [4]

6 Let

$$I_n = \int_0^1 (1 + x^2)^{-n}\,dx,$$

where $n \geqslant 1$. By considering $\frac{d}{dx}\left(x(1 + x^2)^{-n}\right)$, or otherwise, prove that

$$2nI_{n+1} = (2n - 1)I_n + 2^{-n}.$$ [5]

Deduce that $I_3 = \frac{3}{32}\pi + \frac{1}{4}$. [3]

7 Write down an expression in terms of z and N for the sum of the series

$$\sum_{n=1}^{N} 2^{-n} z^n.$$ [2]

Use de Moivre's theorem to deduce that

$$\sum_{n=1}^{10} 2^{-n} \sin\left(\tfrac{1}{10}n\pi\right) = \frac{1025 \sin\left(\frac{1}{10}\pi\right)}{2560 - 2048 \cos\left(\frac{1}{10}\pi\right)}.$$ [6]

8 Find the coordinates of the centroid of the finite region bounded by the x-axis and the curve whose equation is

$$y = x^2(1 - x).$$ [7]

Deduce the coordinates of the centroid of the finite region bounded by the x-axis and the curve whose equation is

$$y = x(1 - x)^2.$$ [2]

9 The planes Π_1 and Π_2 have vector equations

$$\mathbf{r} = \lambda_1(\mathbf{i} + \mathbf{j} - \mathbf{k}) + \mu_1(2\mathbf{i} - \mathbf{j} + \mathbf{k}) \quad \text{and} \quad \mathbf{r} = \lambda_2(\mathbf{i} + 2\mathbf{j} + \mathbf{k}) + \mu_2(3\mathbf{i} + \mathbf{j} - \mathbf{k})$$

respectively. The line l passes through the point with position vector $4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$ and is parallel to both Π_1 and Π_2. Find a vector equation for l. [6]

Find also the shortest distance between l and the line of intersection of Π_1 and Π_2. [4]

10 It is given that the eigenvalues of the matrix **M**, where

$$\mathbf{M} = \begin{pmatrix} 4 & 1 & -1 \\ -4 & -1 & 4 \\ 0 & -1 & 5 \end{pmatrix},$$

are 1, 3, 4. Find a set of corresponding eigenvectors. [4]

Write down a matrix **P** and a diagonal matrix **D** such that

$$\mathbf{M}^n = \mathbf{PDP}^{-1},$$

where n is a positive integer. [2]

Find $\mathbf{P}^{-1}$ and deduce that

$$\lim_{n\to\infty} 4^{-n}\mathbf{M}^n = \begin{pmatrix} -\frac{1}{3} & 0 & -\frac{1}{3} \\ \frac{4}{3} & 0 & \frac{4}{3} \\ \frac{4}{3} & 0 & \frac{4}{3} \end{pmatrix}.$$ [5]

11 Find the rank of the matrix **A**, where

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 4 & 3 & 5 & 16 \\ 6 & 6 & 13 & 13 \\ 14 & 12 & 23 & 45 \end{pmatrix}.$$ [3]

Find vectors $\mathbf{x}_0$ and $\mathbf{e}$ such that any solution of the equation

$$\mathbf{Ax} = \begin{pmatrix} 0 \\ 2 \\ -1 \\ 3 \end{pmatrix} \qquad (*)$$

can be expressed in the form $\mathbf{x}_0 + \lambda\mathbf{e}$, where $\lambda \in \mathbb{R}$. [5]

Hence show that there is no vector which satisfies (*) and has all its elements positive. [3]

12 Answer only **one** of the following two alternatives.

EITHER

Show that $\left(n + \frac{1}{2}\right)^3 - \left(n - \frac{1}{2}\right)^3 \equiv 3n^2 + \frac{1}{4}$. [1]

Use this result to prove that $\sum_{n=1}^{N} n^2 = \frac{1}{6}N(N+1)(2N+1)$. [2]

The sums S, T and U are defined as follows:

$$S = 1^2 + 2^2 + 3^2 + 4^2 + \ldots + (2N)^2 + (2N+1)^2,$$
$$T = 1^2 + 3^2 + 5^2 + 7^2 + \ldots + (2N-1)^2 + (2N+1)^2,$$
$$U = 1^2 - 2^2 + 3^2 - 4^2 + \ldots - (2N)^2 + (2N+1)^2.$$

Find and simplify expressions in terms of N for each of S, T and U. [5]

Hence

(i) describe the behaviour of $\dfrac{S}{T}$ as $N \to \infty$, [1]

(ii) prove that if $\dfrac{S}{U}$ is an integer then $\dfrac{T}{U}$ is an integer. [3]

OR

The curves C_1 and C_2 have polar equations

$$r = 4\cos\theta \qquad \text{and} \qquad r = 1 + \cos\theta$$

respectively, where $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$.

(i) Show that C_1 and C_2 meet at the points $A\left(\frac{4}{3}, \alpha\right)$ and $B\left(\frac{4}{3}, -\alpha\right)$, where α is the acute angle such that $\cos\alpha = \frac{1}{3}$. [2]

(ii) In a single diagram, draw sketch graphs of C_1 and C_2. [3]

(iii) Show that the area of the region bounded by the arcs OA and OB of C_1, and the arc AB of C_2, is

$$4\pi - \frac{1}{3}\sqrt{2} - \frac{13}{2}\alpha.$$ [7]

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